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The Friction Factor–Reynolds Number Relation for the Steady Flow of Pseudoplastic Fluids Through Rectangular Ducts

Part I. Theory

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A finite difference technique is used to calculate the velocity profile, maximum shear stress, and friction factor-Reynolds number product for a power-law, pseudoplastic fluid flowing through a cylindrical duct of rectangular cross section. It is assumed that the motion is rectilinear. A very simple friction factor-Reynolds number correlation is obtained for the special case of a square duct.

Scientific interest in the flow of non-Newtonian fluids has increased steadily in the last few decades. This growth of interest has been motivated by technical requirements in industry and by the desire of rheologists to fill the gap in knowledge between the classical theories of Hookean elasticity and Newtonian flow. Although some very general formulations of the equations of motion for non-Newtonian fluids have been presented in the literature, the physical systems for which these equations have been solved to yield results that can be compared with experimental measurements have been characterized by a high degree of geometric symmetry. In these systems, which include the pipe, the circular annulus, parallel plates, rotating concentric cylinders, and approximately the cone and plate, there exist coordinate systems such that there is only one nonzero component of velocity and the strain-rate tensor has only two nonzero elements. Since the irrotational strain-rate tensor is symmetric, the constituitive equations relating the corresponding elements of the stress and strain-rate tensors reduce to rather simple expressions.

Hence, the problem becomes one of solving a nonlinear ordinary differential equation for which an analytical solution can often be obtained. The square duct, on the other hand, possesses less symmetry than the previously mentioned systems, and it is impossible to find a coordinate system for which there are fewer than four nonzero elements in the stress or strain-rate tensors. Even if it is reasonable to assume that there is only one nonzero component of velocity, the analytical problem involves the solution of a nonlinear partial differential equation for which analytical solutions can be obtained only in very special cases.

To further complicate the situation, it is now known that many non-Newtonian fluids possess elastic as well as viscous properties when subjected to strain. Although this has no effect on velocity profiles existing in the very simple systems mentioned earlier, it may have a profound effect on the velocity field existing in a rectangular duct. In particular, the normal stresses generated in a viscoelastic fluid are expected to generate a secondary flow which is transverse to the main flow along the axis of the duct. Judging from theoretical results calculated for the ellipse (*8*), one can speculate that particle trajectories in the square duct might be helical with the secondary flow directed inward along the diagonals connecting opposite corners and outward along the center lines perpendicular to the surfaces. To the authors' knowledge, such flows have not been observed.

For one relatively simple viscoelastic model, the three-constant Oldroyd model proposed by Williams and Bird (*10*), normal stresses perpendicular to the direction of flow are equal. In this case, there will be no secondary flow. Study of secondary flow in a square duct may provide a reasonably sensitive method for assessing the degree to which normal stresses are equal. Visual observations seem to indicate that for one fluid (sodium carboxymethylcellulose) transverse components of velocity are small relative to the main axial component. If this is true, there exists a good possibility of obtaining an accurate description of the velocity for fairly complex models by using a perturbation method. However, this will be possible only after it has been established that very accurate solutions can be obtained for the unperturbed axial velocity field because calculation of the driving force for the secondary flow involves several differentiations of the axial velocity.

Prior work on the unperturbed problem includes a paper by Schechter (*9*), who demonstrated that a variational method can be employed when the power law model is used. Although this is an elegant method, its use is impractical when highly accurate velocity profiles are required. It is also of limited applicability because there exist many fluids for which a variational principle cannot be written. Clark and Kays (*2*) used an overrelaxation (O.R.) technique to solve the problem for a Newtonian model, but the method has not been applied to a non-Newtonian fluid. The purpose of this paper is to establish that very accurate values can be obtained for the unperturbed axial velocity profiles with an O.R. method. A friction factor-Reynolds number correlation based on the unperturbed profiles is obtained also for power law fluids. In part II of this paper, a comparison between the calculated and measured friction factor-Reynolds number correlation will be presented for aqueous solutions of CMC.

STATEMENT OF THE PROBLEM

The basic form of the equations of motion and the continuity equation are well established (*1*), but there still exists considerable uncertainty about the proper statement of the constituitive relation which defines the elements of the stress tensor in terms of the elements of the strain-rate tensor. Some feeling for the state of the art can be gleaned from recently published books on rheology (*4, 7*).

Consider a rectangular duct with Cartesian coordinates oriented as shown in Figure 1. For the unperturbed flow, there is only one nonzero component of velocity V_z', and

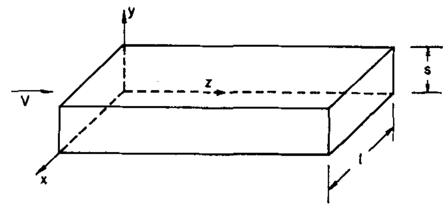

Fig. 1. Flow in a rectangular duct.

the equations of motion reduce to a single nonlinear partial differential equation of the form

$$\frac{\partial}{\partial x}\left(u\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial V}{\partial y}\right)+\frac{f_D N_{ReD}}{2}=0 \quad (1)$$

in which all of the variables and parameters are dimensionless. The function u is a variable viscosity which is defined in terms of K, the second invariant of the irrotational strain-rate tensor:

$$K=\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2 \quad (2)$$

Equation (1) must be solved subject to the boundary conditions

$$V(0,y)=V(1,y)=V(x,0)=V(x,S)=0 \quad (3)$$

and the constraining equation

$$\int_0^S\int_0^1 V(x,y)\,dx\,dy=1 \quad (4)$$

Equation (4) can be regarded as indirectly relating the value of $f_D N_{ReD}$ to other parameters, such as S and n, which appear in the constituitive equation.

The power law model was used in this work to describe the stress strain-rate relation for pseudoplastic fluids:

$$u=K^{\frac{n-1}{2}} \quad (5)$$

This model has two serious theoretical defects. The local viscosity approaches infinity as K approaches zero and approaches zero as K becomes very large. In spite of the fact that no real fluid can have these properties, the power law model has proven to be adequate for describing the flow of many fluids through circular pipes. Since it is not anticipated that there will be a great difference between the unperturbed flow in a duct and flow through a circular pipe, it is reasonable to assume that the power law model will be adequate. On the other hand, Fredrickson (*6*) has found that it is inadequate to interrelate the flow in annuli and circular pipes.

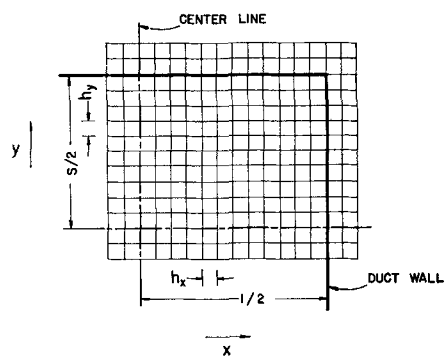

Fig. 2. Relaxation mesh in a quadrant of the rectangular duct.

NUMERICAL SOLUTIONS OF THE EQUATIONS

The O.R. method, like all finite-difference methods, is characterized by the replacement of continuous functions by discrete distributions defined at the nodes of a rectangular mesh superimposed on one quadrant of a cross section of the duct as shown in Figure 2. Note that the mesh extends for two intervals beyond the boundaries of the quadrant. The mesh shown in Figure 2 has twelve increments between the center line and the wall in the x direction and ten increments in the y direction. Such a mesh will be referred to as a 12×10 mesh. The mesh sizes actually employed for computations were 10×10, 20×20, and 40×40. The nodes in the neighborhood of a typical point (x_0, y_0) will be designated as shown in Figure 3. The values of V and u in the neighborhood of (x_0, y_0) are denoted by subscripted variables, for example, $V_{1,-1}$ or $u_{1,-1}$. The first index gives the position relative to (x_0, y_0) in the x direction, and the second index gives the position in the y direction.

The O.R. method is an iteration procedure requiring initial estimates for V and u at each mesh node. An initial estimate for $f_D N_{ReD}$ must also be selected. It is convenient to first solve the problem for the Newtonian case, $n = 1$, and then solve the problem for successively smaller values of n with the solution for one case as the initial estimate for the next case.

The iteration formulas used to improve successive estimates of V at the mesh nodes can be derived by first writing Equation (1) in the form

$$\nabla^2(uV) + u\nabla^2 V - V\,\nabla^2 u + f_D N_{ReD} = 0 \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{7}$$

There are two reasonable finite-difference approximations for the Laplacian (5):

$$\nabla^2 V(x_0, y_0) \cong \nabla^2_{h5} V = \frac{V_{1,0} + V_{-1,0} - 2V_{0,0}}{h_x^2} = \frac{V_{0,1} + V_{0,-1} - 2V_{0,0}}{h_y^2} \tag{8}$$

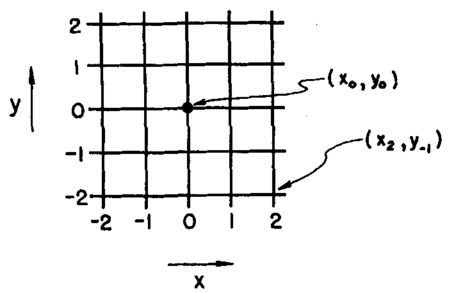

Fig. 3. Designation of mesh nodes relative to a central point.

and

$$\nabla^2 V(x_0, y_0) \cong \nabla^2_{h9} V = \frac{16(V_{1,0} + V_{-1,0}) - V_{2,0} - V_{-2,0} - 30V_{0,0}}{12h_x^2} + \frac{16(V_{0,1} + V_{0,-1}) - V_{0,2} - V_{0,-2} - 30V_{0,0}}{12h_y^2} \tag{9}$$

where h_x and h_y are the distances between adjacent mesh nodes in the x and y directions, respectively. The factors entering into the selection of the approximation to be used are accuracy, stability, rate of convergence, and complexity. These factors are interrelated and vary with the value of n and the mesh size. Numerical experiments in the solution of Equation (1) indicated that the stability and convergence characteristics of a finite-difference equation based on the nine-point approximation [Equation (9)] do not differ significantly from those based on the five-point formula [Equation (8)]. The finite-difference equation based on the nine-point approximation did yield significantly more accurate results for a given mesh size and thus was selected for this investigation. Replacing the Laplacian in Equation (6) by its finite-difference analogue and solving for $V_{0,0}$ one gets

$$V_{0,0} = \frac{16[(u_{1,0} + u_{0,0})V_{1,0} + (u_{-1,0} + u_{0,0})V_{-1,0}] - (u_{2,0} + u_{0,0})V_{2,0} - (u_{-2,0} + u_{0,0})V_{-2,0} + (h_x^2/h_y^2)\{16[(u_{0,1} + u_{0,0})V_{0,1} + (u_{0,-1} + u_{0,0})V_{0,-1}] - (u_{0,2} + u_{0,0})V_{0,2} - (u_{0,-2} + u_{0,0})V_{0,-2}\} + 12h_x^2 f_D Re_D}{16(u_{1,0} + u_{-1,0}) - u_{2,0} - u_{-2,0} + 30u_{0,0} + (h_x^2/h_y^2)\{16(u_{0,1} + u_{0,-1}) - u_{0,2} - u_{0,-2} + 30u_{0,0}\}} \tag{10}$$

Equation (10) is the basic iteration formula used in the numerical computations.

Convergence of the iteration process can be accelerated by employing the formula

$$V_{0,0}^{i} = V_{0,0}^{i-1} + W\,[(V_{0,0})_c - V_{0,0}^{i-1}] \tag{11}$$

where the superscripts indicate the iteration number, $(V_{0,0})_c$ is the value calculated by Equation (10) from values of u and V determined in the $(i-1)$th iteration, and W is the overrelaxation factor. One application of Equation (11) at each mesh node on the center lines and in the interior of the quadrant of the duct constitutes one iteration of the velocity distribution.

After six to twelve iterations of the velocity distribution, improved values of the reduced viscosity coefficient were computed. The finite-difference analogue

$$u_{0,0} = \left\{ \frac{[8\,(V_{1,0} - V_{-1,0}) - V_{2,0} + V_{-2,0}]^2}{144\,h_x^2} + \frac{[8\,(V_{1,0} - V_{-1,0}) - V_{0,2} + V_{0,-2}]^2}{144\,h_y^2} \right\}^{(n-1)/2} \tag{12}$$

of Equation (5) was used for this purpose. The application of Equation (12) at each mesh node on the boundaries and in the interior of the quadrant constitutes one improvement of the viscosity distribution. Since the invariant K is zero at the center and at the corners of the duct, the formula

$$u = K^{(n-1)/2} \qquad n \leqq 1 \tag{5}$$

cannot be applied at these points. If the minimum value of K is set equal to some small number, say 10^{-6}, the velocity profile and the friction factor-Reynolds number are unaffected for all practical purposes. The results of numerical experiments indicate that the solutions to the problem are insensitive to the minimum value of K provided that it is less than 10^{-4}.

The application of Equations (10) and (12) in the vicinity of the boundaries of the quadrant requires values of V and u at mesh nodes outside of the quadrant. The symmetry of the system is such that values of V and u exterior to the quadrant, but interior to the duct, may be obtained by simply reflecting values across the center lines. The synthetic values required exterior to the duct can be obtained by finite-difference extrapolation formulas. A suitable formula for the synthetic values of the velocity in the first column of mesh nodes outside of the duct wall parallel to the y axis is

$$V_{k+1,j} = 5V^{0}_{k,j} - 10\,V_{k-1,j} + 10\,V_{k-2,j} - 5V_{k-3,j} + V_{k-4,j} \tag{13}$$

where k is the x index corresponding to the wall of the duct (5). The formula for the second column of mesh nodes exterior to the duct is

$$V_{k+2,j} = 15V^{0}_{k,j} - 40V_{k-1,j} + 45V_{k-2,j} - 24V_{k-3,j} + 5V_{k-4,j} \tag{14}$$

The other extrapolation formulas required are similar to the two above. After each improvement of V and u in the quadrant, the values exterior to the quadrant were corrected by extrapolation and reflection.

Successive iterations were continued until each of the velocities computed immediately after an improvement of the viscosity distribution agreed to within some preselected value with the corresponding V computed immediately before the viscosity improvement. In this investigation, the number selected was 10^{-6}.

In general, the iteration process will converge to a velocity distribution having an average value $\overline{V}$ not equal to unity as required by Equation (4). The discrepancy is due to the error in the initial estimate of $f_D N_{ReD}$. A method of correcting the error may be devised by dividing each term in Equation (1) by $\overline{V}^n$, by writing the result in the form

$$\frac{\partial}{\partial x}\left[\frac{u}{\overline{V}^{n-1}}\frac{\partial}{\partial x}\left(\frac{V}{\overline{V}}\right)\right] + \frac{\partial}{\partial y}\left[\frac{u}{\overline{V}^{n-1}}\frac{\partial}{\partial y}\left(\frac{V}{\overline{V}}\right)\right] + \frac{f_D\,N_{ReD}}{2\overline{V}^{\,n}} = 0 \tag{15}$$

and by noting that

$$\frac{u}{\overline{V}^{n-1}} = \left\{\left[\frac{\partial}{\partial x}\left(\frac{V}{\overline{V}}\right)\right]^2 + \left[\frac{\partial}{\partial y}\left(\frac{V}{\overline{V}}\right)\right]^2\right\}^{(n-1)/2} \tag{16}$$

These two relations indicate that the changes

$$\frac{V}{\overline{V}} \rightarrow V \tag{17}$$

$$\frac{u}{\overline{V}^{n-1}} \rightarrow u \tag{18}$$

$$\frac{f_D\,N_{ReD}}{\overline{V}^{n}} \rightarrow f_D\,N_{ReD} \tag{19}$$

result in a new velocity distribution having an average value of unity and new values of u and $f_D\,N_{ReD}$ which together with the new values of V satisfy Equations (1), (4), and (5). Although the velocity distribution need be normalized only after the iteration process has converged, normalizing after every fifteen or twenty iterations appears to increase the rate of convergence. Simpson's rule extended

$$\int_{x_0}^{x_2} y\,dx \sim \frac{h}{90}\left[114y_1 + 34(y_0 + y_2) - (y_{-1} + y_3)\right] \tag{20}$$

was applied to compute $\overline{V}$.

It is well known that the accuracy of an overrelaxation solution is greater for a fine mesh than for a coarse mesh (5). The rate of convergence, on the other hand, is greater for a coarse mesh than for a fine mesh. The number of computations required to obtain a given accuracy can be reduced by solving the problem first with a coarse mesh and then solving it with a finer mesh with the results of the first solution used as initial estimates for the second solution.

The greatest difficulty encountered in applying the overrelaxation method is achieving a stable iteration process while maintaining a reasonable rate of convergence. Stability and rate of convergence are functions of the overrelaxation factor, the mesh size, the power law parameter, the nature of the estimated velocity profile, the number of iterations of the velocity profile between improvements of the viscosity distribution, and the duct aspect ratio. Although the determination of quantitative relations is beyond the scope of this investigation, several qualitative observations can be made.

The overrelaxation factor is the key quantity. If W is small enough, the iterative process is stable regardless of other factors adversely affecting stability. Unfortunately, the process converges slowly for relatively small values of W. All other factors being equal, a larger value of W should be used for a fine mesh than for a coarse mesh. Because of the nonlinear character of Equations (10) and (12), stability was found to depend on the error spectrum of the initial estimate for the velocity profile. A considerable increase in stability was achieved by smoothing the initial estimate by using $W = 0.5$ for several iterations and improving the viscosity coefficients after each iteration. Table 1 contains recommended values of W for various values of n, S, and mesh size. The values of W listed are probably smaller than the optimum values.

Both stability and the rate of convergence decrease rapidly as the value of the power law parameter n is decreased. For n equal to or greater than 0.6, a convenient value of W could always be found. For n equal to 0.5 or less, stability was a serious problem. As indicated in Table 1, relatively small values of W were required for small values of n. Several hundred iterations were required for the process to converge for n equal to 0.4.

After the initial smoothing, the viscosity coefficients were recalculated after every sixth iteration on the velocity for n equal to 0.5 or greater. For n equal to 0.4, less

Table 1. Recommended Values of the Overrelaxation Factor

n	S	Mesh size	W
$n \geqq 0.6$	1	10×10	1.7
	1	20×20	1.9
	$S < 0.75$	10×10	1.5
	$S < 0.75$	20×20	1.8
0.5	1	10×10	1.6
	1	20×20	1.8
0.4	1	10×10	1.2
	1	20×20	1.7

TABLE 2. THE FRICTION FACTOR-REYNOLDS NUMBER PRODUCT FOR THE RECTANGULAR DUCT

n	S	$f_D N_{ReD}$*	$f_D N_{ReD}$†
1.0	1	56.91	57.08
0.9	1	47.62	—
0.8	1	39.66	—
0.75	1	36.22	36.34
0.7	1	33.07	—
0.6	1	27.53	—
0.5	1	22.89	23.02
0.4	1	18.97	—
1.0	0.5	139.9	140.4
0.75	0.75	48.05	47.51
0.75	0.5	78.83	105.0**

* Computed by overrelaxation method.
† Computed by the variational method.
** This value has been found to be in error.

than ten velocity iterations between improvements of the viscosity coefficients resulted in instability

A Fortran program employing the overrelaxation method was run on the Control Data Corporation 1604 computer at The University of Texas Computation Center. The values of the friction factor-Reynolds number product computed are given in Table 2, together with some of the values computed by Schechter (9) by the variational method.

To facilitate comparison of experimental and theoretical results, the equation

$$f_D\, N_{ReD} = 7.4942 \left(\frac{1.7330}{n} + 5.8606 \right)^n \tag{21}$$

was devised for the square duct, that is S equal to 1. The constants in Equation (21) were obtained by a least-square fit of the eight values of $f_D N_{ReD}$ in Table 2. Values of $f_D N_{ReD}$ computed by the regression equation agree to four significant figures with the values in Table 2.

The maximum stress in the duct is required in the analysis of experimental data. The point of maximum stress corresponds to the point at which the dimensionless viscosity coefficient is a minimum. For the rectangular duct, this point occurs at the wall on one of the center lines. The relationship between the maximum dimensionless stress τ and u is

$$\tau_{\max} = (u_{\min})^{n/(n-1)} \tag{22}$$

where

$$\tau_{\max} = \frac{L^n}{m V^n_{\text{avg}}} (\tau_{xz})_{\max} \tag{23}$$

It is convenient to tabulate the ratio $\tau_{\max}/f_D N_{ReD}$, as is done in Table 3, since this ratio varies only slightly with n.

The accuracy of the values obtained by an overrelaxation method is difficult to estimate. There are two methods of estimating the accuracy of the computed value of $f_D N_{ReD}$. Analytical solutions can be derived for laminar flow

TABLE 3. MAXIMUM STRESS IN THE SQUARE DUCT

n	$\dfrac{\tau_{\max}}{f_D N_{ReD}}$	n	$\dfrac{\tau_{\max}}{f_D N_{ReD}}$
1	0.1688	0.7	0.1629
0.9	0.1670	0.6	0.1605
0.8	0.1651	0.5	0.1579
0.75	0.1640	0.4	0.1550

of a Newtonian fluid in a rectangular duct (3) and for the flow of a power law fluid between parallel plates:

rectangular duct

$$f_D N_{ReD} = 24.0 \Big/ \left(1 - \frac{192}{S\pi^5} \sum_{k=1,3,5,\dots} \frac{\tanh (k\pi S/2)}{k^5} \right) \tag{24}$$

parallel plates

$$f_D N_{ReD} = 4 \left(\frac{2}{n} + 4 \right)^n \tag{25}$$

The second method of estimating the error is to solve a given problem with one mesh size and then solve the same problem with a mesh having four times as many nodes as the first mesh. The error in the second solution is probably less than the deviation between the two solutions.

Comparison of the numerical and analytical values of $f_D\, N_{ReD}$ for a Newtonian fluid when $S = 1.0$ and 0.5 indicates that the computational errors are 0.0023 and 0.0050%, respectively. The numerically computed vlaue for a non-Newtonian fluid with $n = 0.75$ and $S = 500$ is 0.051% different than the analytical value for two flat plates. Changing the mesh size from 10 × 10 to 20 × 20 for $n = 0.5$, 0.6, and 0.7 results in changes in $f_D\, N_{ReD}$ of 0.04, 0.03, and 0.02%, respectively. Going from a 20 × 20 mesh to a 40 × 40 mesh for $n = 0.75$ and $S = 0.75$ produces a change in $f_D\, N_{ReD}$ of 0.002%. On the basis of the above information, it is estimated that the maximum error in the O.R. values of $f_D\, N_{ReD}$ recorded in Table 2 is no larger than 0.1%. Even more important is the fact that very great accuracy can be attained by using a fine mesh. For example, the error in the $f_D N_{ReD}$ for a 40 × 40 mesh when $n = 0.75$ and $S = 0.75$ is probably about 0.001%. A few solutions were obtained for an 80 × 80 mesh, and these gave velocity profiles which could be differentiated numerically several times. In comparison, analytical differentiation of velocity profiles obtained by the variational method produced results that were very erratic. An attempt to improve the accuracy of the variational results by including more terms in the trial function was not successful because the amount of machine time increased very rapidly as the number of terms increased. Hence, in those cases where highly accurate velocity profiles are required, the finite-difference method seems to possess definite advantages over the variational method.

NOTATION

- a = height of the rectangular duct, cm.
- f_D = friction factor for flow in a rectangular duct, dimensionless
- h_x, h_y = dimensionless distances between adjacent mesh nodes in the x and y directions, respectively
- K = dimensionless invariant defined by Equation (2)
- L = width of the rectangular duct, cm.
- m = parameter in the power law model, dynes sec.^{n}/(sq. cm.)
- n = parameter in the power law model, dimensionless
- P = pressure, dynes/(sq. cm.)
- N_{ReD} = $L^n \rho V^{2-n}_{\text{avg}}/m$, Reynolds number for flow in a rectangular duct, dimensionless
- S = a/L, aspect ratio of the rectangular duct, dimensionless
- u = dimensionless local viscosity defined by Equation (5)
- $u_{0,0}, u_{2,-1}$, etc. = values of u at the mesh nodes indicated in Figure 3, dimensionless
- V = V'_z/V_{avg}, dimensionless fluid velocity
- V'_z = fluid velocity at a point in the rectangular duct, cm./sec.

V_{avg} = average fluid velocity in a duct, cm./sec.

$V_{0,0}, V_{2,-1}$, etc. = values of V at the mesh nodes indicated in Figure 3, dimensionless

$\overline{V}$ = computed quantity used in Equations (15) through (19), dimensionless

W = overrelaxation factor, dimensionless

x', y', z' = rectangular Cartesian coordinates, cm.

x, y, z = dimensionless coordinates, such as $x = x'/L$

x_0, y_{-1}, etc. = values of x and y at particular mesh nodes as indicated in Figure 3, dimensionless

∇^2 = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}$, dimensionless

τ_{max} = maximum shear stress exerted on the fluid in a given experiment, dynes/(sq. cm.)

ρ = fluid density, g./cc.

μ = $m\left[\left(\frac{\partial v'}{\partial x'}\right)^2 + \left(\frac{\partial v'}{\partial y'}\right)^2\right]^{n-1/2}$, local viscosity at a point in the square duct, dyne sec./(sq. cm.)

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Part II. Experimental Results

The friction factor and Reynolds number were measured for sodium carboxymethylcellulose (cmc) flowing through a rectangular duct. Measured values were compared with values calculated in Part I of this paper. Both a pipe viscometer and a Couette viscometer were used to evaluate the rheological parameters for the fluids. Calculations were based on a power law model. The agreement between theoretical and experimental values was excellent.

Two papers have recently been presented discussing the laminar flow of pseudoplastic fluids through long rectangular ducts. Schechter (3) devised a variational principle which he used to obtain approximate solutions for the equation of motion, while Wheeler and Wissler (5) used an iterative finite-difference method. In both papers, velocity profiles and the friction factor-Reynolds number product were calculated for several rectangular cross sections and a range of fluid properties.

One can anticipate that experimentally obtained data might differ significantly from the corresponding calculated data for one or more of the following reasons: the rheological model is inadequate to describe the fluid, the approximate solutions are not sufficiently accurate, or experimental errors are too large. Clearly, computational and experimental errors must be kept small if one hopes to draw any conclusions about the adequacy of the model.

The principal reason for suspecting that the rheological model might be inadequate is that many pseudoplastic fluids are also viscoelastic. Consequently, normal stresses, as well as the usual shear stresses, are developed when the fluid is pumped through a square duct, and this can lead to a transverse secondary flow which is superimposed on the main axial flow. The transverse secondary flow is manifest in a modified main flow due to both changes in the shear stresses and the existence of nonzero acceleration in the equation of motion.

In this paper the authors are presenting some reasonably accurate experimental data which can be used to check the validity of neglecting secondary flow, at least as a first approximation. The data were consolidated into pairs of dimensionless numbers consisting of the friction factor and the Reynolds number. Admittedly, the friction factor-Reynolds number correlation is not as sensitive to details of the rheological model as are the velocity profiles because the Reynolds number depends only on the average velocity. It is possible that small but theoretically significant deviations from the predicted velocity profiles are not reflected to a detectable extent in the observed correlation. However, large differences between the true velocity profile and the predicted profile should be detectable in the friction factor correlation.

Some idea of the inaccuracy of the computed results can be obtained by comparing corresponding results obtained by two independent workers. Although completely

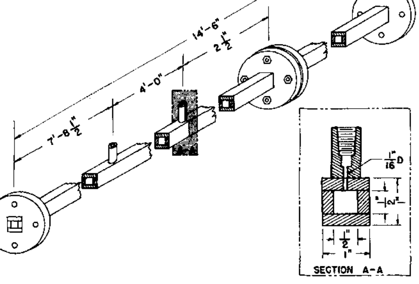

Fig. 1. Duct test section.